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Vibration of Dynamic Systems Under Cyclostationary Excitations

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A special type of nonstationary, random excitation, called *cyclostationary*, is studied. The main characteristic of this type of excitation is that its statistical properties (e.g., the rms) vary periodically in time in contrast to a traditional, random stationary model, which assumes constant statistical properties. Many engineering structures, such as a submarine propeller, a turbine blade, and an internal combustion engine, are subjected to this type of excitation. A method for modeling the excitation and for calculating the response of such systems is presented. It is shown that the road excitation on a vehicle driven on a road made of concrete slabs, whose length is constant, is cyclostationary. A cyclostationary model can yield considerably more accurate estimates of the rms of the response as compared to a traditional stationary model in this problem.

Nomenclature

$A(t)$, $B(t)$	=	zero-mean stationary random processes
c	=	damping coefficient
d	=	length of a slab
$E[\,]$	=	expected value of a random variable
$f(\,)$	=	probability density function
H_i	=	height of the ith slab
$\boldsymbol{H}(\omega)$	=	matrix of frequency response functions
$\boldsymbol{h}(t)$	=	matrix of impulse response functions
I	=	set of integers
i	=	index
$\boldsymbol{K}_{YY}(t_1, t_2)$	=	covariance matrix of the excitation vector
$\boldsymbol{K}_{ZZ}(t_1, t_2)$	=	covariance matrix of the response vector
k	=	spring constant
m	=	mass of the road vehicle
n	=	integer
$P(t)$, $Q(t)$	=	CS processes
pulse( )	=	pulse function
$R_{uu}(t_1, t_2)$	=	autocorrelation of the velocity in the wake of a ship
$R_{YY}(t_1, t_2)$	=	autocorrelation of the road excitation
$\boldsymbol{R}_{YY}(t_1, t_2)$	=	correlation matrix of the excitation vector
$\boldsymbol{R}_{ZZ}(t_1, t_2)$	=	correlation matrix of the response vector
$\boldsymbol{R}^S_{ZZ}(t_1, t_2)$	=	correlation matrix of the response vector of the equivalent stationary random processes
$\boldsymbol{r}$	=	position vector
$\boldsymbol{r}_{YY}(n, \tau)$	=	cyclic correlation matrix of the excitation vector
$\boldsymbol{r}_{ZZ}(n, \tau)$	=	cyclic correlation matrix of the response vector
$S(t)$	=	resultant process after phase randomization
$\boldsymbol{S}_{YY}(n, \omega)$	=	matrix of cyclic cross-spectral density of the excitation vector
$\boldsymbol{S}_{ZZ}(n, \omega)$	=	matrix of cyclic cross-spectral density of the response vector
T	=	time period
t	=	time
$u(\boldsymbol{r}, t)$	=	horizontal component of the velocity
$\boldsymbol{u}(\boldsymbol{r}, t)$	=	velocity field in the plane of a propeller
u_0	=	ship velocity
V	=	velocity of the road vehicle
$w(\boldsymbol{r}, t)$	=	wake fraction
$Y(t)$	=	excitation caused by road roughness
$\boldsymbol{Y}(t)$	=	excitation vector (size $= m \times 1$)
$\boldsymbol{Z}(t)$	=	response vector (size $= p \times 1$)
Δ	=	distance between two points where the autocorrelation is being calculated
$\eta(t)$	=	mean of the excitation vector to a dynamic system
σ	=	standard deviation
τ	=	time difference $(t_1 - t_2)$
ϕ_1, ϕ_2	=	random phase angles
ω	=	frequency
ω_e	=	fundamental frequency
$'$	=	denotes transpose of a vector or matrix

Introduction

CYCLOSTATIONARY (CS) random processes are nonstationary processes, whose statistical properties vary periodically in time. They are fundamentally different than stationary random processes. Figures 1 and 2 show the difference between the sample path of a CS process and the sample path of a stationary process. Figure 2 shows that the signal fluctuates more vigorously at $t = 0$, 5, 10, and 15 s than at $t = 2.5$, 7.5, and 12.5 s. The change in the intensity of the signal from a maximum to the next minimum (and a minimum to the next maximum) takes place in equal intervals of 2.5 s, approximately. This causes a periodic variation in the statistics of the signal. This time variation in the statistics of the signal is not seen in Fig. 1, which is the sample path of a stationary random process.

Many stochastic processes encountered in real life exhibit an inherent periodicity in the variation of their statistical properties and should be modeled as CS processes. This periodic variation in the statistical properties can be caused by the following:

1) The first is repetitive operation, such as sampling and scanning on random signals. A common example is the output of a receiver connected to a narrow-beam radar antenna, which rotates in a nonuniform field produced by a stationary signal source.[1]

2) The second is rotation of a component of a structure in a nonuniform flowfield. For example, as the blades of a gas turbine rotate, they encounter a random velocity field, which is caused by the flow distortion as a result of the equally spaced stator blades.[2] The statistical properties of the encountered velocity vary periodically. Lin et al.[3] and Fujimori et al.[4] studied the response of a helicopter blade to turbulence in forward flight using a model whose parameters were periodic, nonstationary random processes.

3) The third is systems subjected to spatially periodic excitation. Dimentberg[5] presented an example of a coal-mine cage traveling

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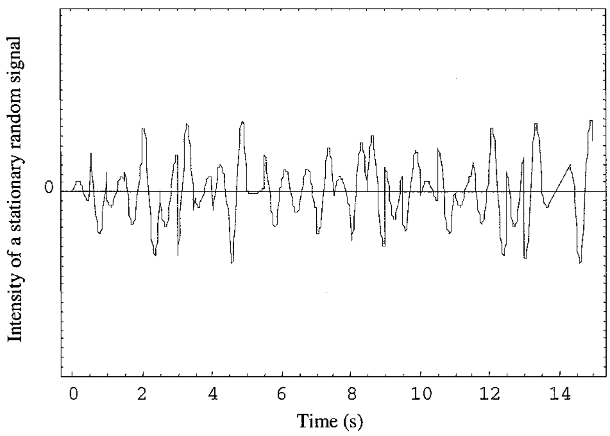

Fig. 1 Sample path of a stationary process.

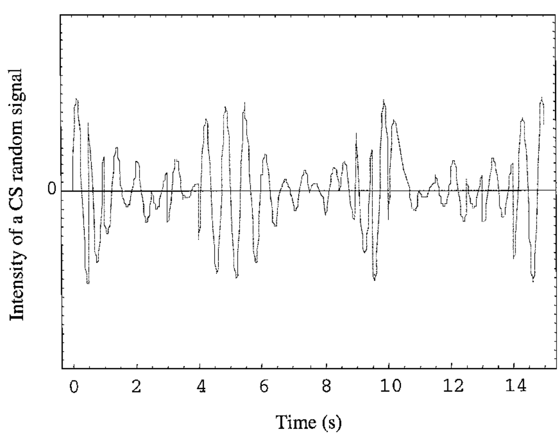

Fig. 2 Sample path of a CS process.

with a constant speed along an elastic cable supported by equally spaced supports. The apparent lateral stiffness of the cable varies periodically along its length, being minimal at midspans and infinity at the supports. Therefore, the traveling cage feels periodic-in-time variations of the stiffness of its suspension.[5] This induces a CS excitation on the coal-mine cage.

Abundant experimental data exist on the characteristics of randomly fluctuating flowfields encountered by turbine rotors, helicopter rotors, and marine propellers. These data show that these flowfields should be modeled as CS random processes because the intensity of random fluctuation of the excitation is important, and it varies significantly in time. Figure 3 shows that the ensemble rms velocity at a fixed point behind a propeller in a wind tunnel varies significantly in time, and it can be as high as 20% of the average velocity.[6] A stationary model of the velocity would assume a constant rms.

For a ship propeller it is recognized that the randomness in the blade excitation, which is caused by the flow distortion from the ship hull, is important. Here we use an example of the flowfield behind a ship hull to explain the characteristics of CS processes. The flow velocity $\boldsymbol{u}(\boldsymbol{r}, t)$ depends on the position vector $\boldsymbol{r}$ and the time t and it fluctuates randomly as a result of turbulence. Because of the presence of hull, the intensity of fluctuation is nonuniform over the propeller disk. For example, the rms velocity and the correlation of the components of the velocity vector vary over the propeller disk. Figure 4 shows a typical wake survey result for the horizontal component of the velocity $u(\boldsymbol{r}, t)$. The labels of the isowake curves specify the wake fraction coefficient $w(\boldsymbol{r}, t) = [1 - u(\boldsymbol{r}, t)/u_0]$. According to this figure, u is small at position A because the hull reduces the velocity at this position, whereas u is almost equal to the ship speed at point B.

In the following we consider only the horizontal velocity component $u[\boldsymbol{r}(t), t]$ at a fixed point on a propeller blade. The position vector $\boldsymbol{r}$ is a function of time because the propeller rotates. Consider

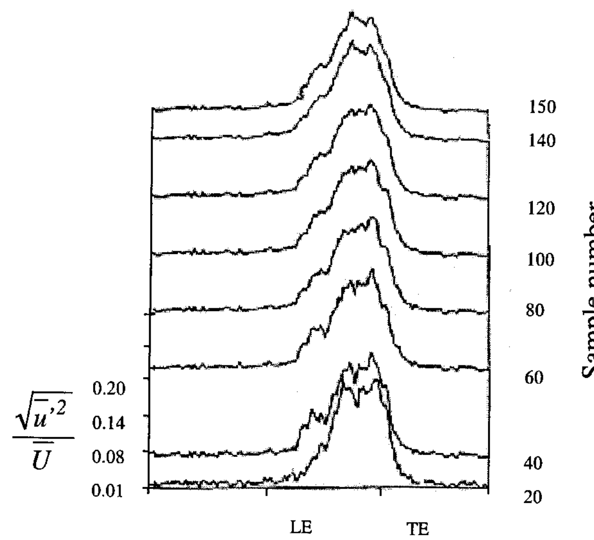

Fig. 3 Ensemble rms velocity at a fixed point behind a propeller. The intensity of random fluctuation is significant and varies in time.

the blade rotating in the flow distorted by the hull. The encountered velocity consists of two parts, the mean velocity (phase locked), which varies cyclically in time, and a random turbulent part. The velocity encountered by a point fixed on a propeller blade is a random process. Figure 5 depicts the autocorrelation of $u(\boldsymbol{r}, t)$ over a period of rotation T. Autocorrelation is the ensemble mean of the product of the values of the velocity $u(\boldsymbol{r}, t_1)$ and $u(\boldsymbol{r}, t_2)$ measured at two time instances t_1 and t_2. In this figure the autocorrelation $R_{uu}(t_1, t_2)$ is presented as a function of t_2 $(= t)$ and $t_1 - t_2$. Consider that at time $t = 0$ the blade is vertical and the tip is at the upper position. In contrast to the case of a stationary process, the autocorrelation $R_{uu}(t_2, t_1 - t_2)$ depends on both the time difference $t_1 - t_2$ and time t_2. The following paragraphs explain the behavior of this function.

First, consider that $t_1 = t_2$ in which case the autocorrelation reduces to the phase-locked mean square. The curve on the plane $t_1 - t_2 = 0$ in Fig. 5 shows the behavior of the autocorrelation. At times $t_2 = 0$ and T the mean square velocity encountered by the blade is large because the blade is in the wake induced by the hull. At $t \approx T/4$ and $3T/4$ the blade is almost out of the boundary layer, and consequently, the mean square velocity it encounters is low.

Second, let $t_2 = 0$. In this case the velocity is evaluated or measured at time equal to zero and some other time instant t_1. The curve on the plane $t_2 = 0$ in Fig. 5 shows the behavior of the autocorrelation in this case. As t_1 increases starting from zero, the autocorrelation decreases because the force is evaluated at two positions of increasing distance. However, as t_1 approaches $T/2$, the force is evaluated at two positions within the hull wake, and therefore, the values are positively correlated. As a result, $R_{uu}(0, T/2)$ is a local maximum. After $t_1 = T/2$ the autocorrelation will initially drop and then increase again as t_1 tends to T. This is because, as t_1 approaches T, the blade initially exits from and then reenters the boundary layer of the ship.

In conclusion, the autocorrelation is a function of both the time instant when the velocity is first measured t_2 and the time lag between the two instances at which the velocity is measured $t_1 - t_2$. This implies that the velocity, the resulting pressure, and the forces are nonstationary processes. If the velocity were stationary, autocorrelation function would not fluctuate with t_2 (Fig. 5). Finally, the observation can be made that the autocorrelation is periodic in time t_2, with period equal to the rotation period T. Whereas nonstationary processes are cyclostationary,[7] the term periodic nonstationary has also been used for these processes. If the flowfield behind a rotating blade is strongly nonuniform, then the autocorrelation is expected to vary significantly with t_2. A cyclostationary model would be more suitable than a stationary in such cases.

CS processes have been studied in depth in signal processing.[1,7,8] However, in mechanics they are rarely used in modeling excitation

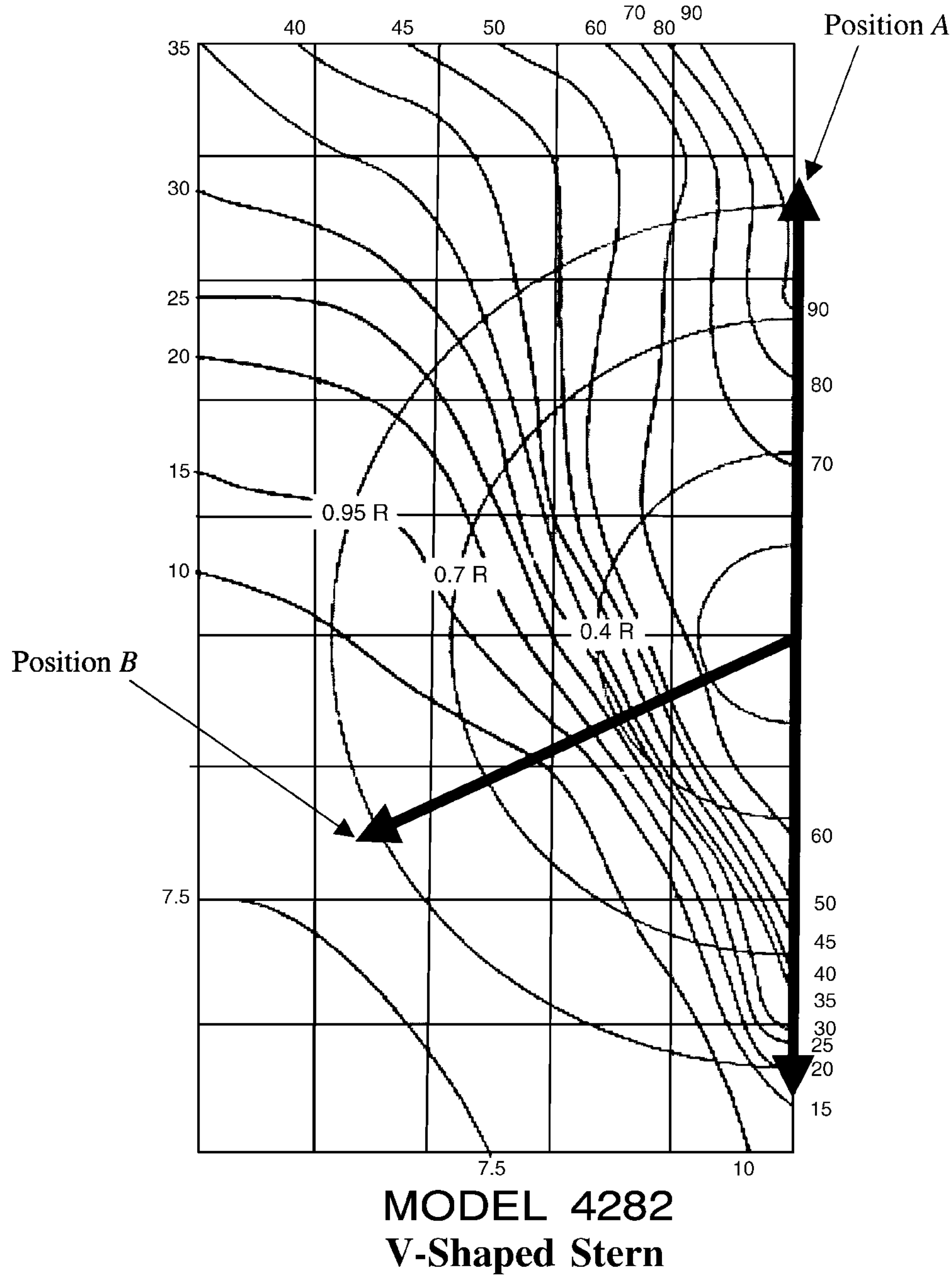

Fig. 4 Wake profile of axial component of velocity $u(r, t)$ behind a ship hull. The contours correspond to constant values of the wake fraction coefficient $w = [1 - u(r, t)/u_0]$, where u_0 is the speed of ship.

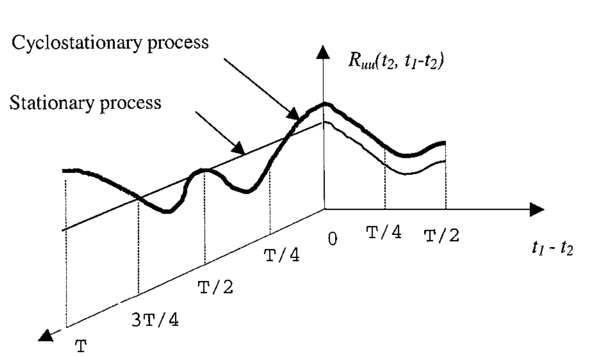

Fig. 5 Autocorrelation $R_{uu}(t_2, t_1 - t_2)$ of the horizontal velocity component $u(r, t)$. The autocorrelation of the CS process varies periodically with t_2, whereas the autocorrelation of a stationary process is constant in t_2.

whose statistics demonstrate periodicity. Such models can be appropriate in problems involving rotating components, such as propeller, turbine, or helicopter blades rotating in turbulent fields.

It has been almost a general trend to stationarize a CS process, which allows us to use available methods for stationary random processes to find the response of a dynamic system. We can stationarize a CS signal as follows:

1) Assume that the CS signal has a random phase angle that is distributed uniformly over the range of 0–2π. For example, each time we measure the pressure time histories on the blades of a turbine, we assume that there is equal chance of starting at an azimuth angle from 0 to 2π.

2) Replace the statistical properties, which are periodic in time, by their temporal average.

However, these methods lose information about the power representation, phase angle, and the time variation of the statistics of the signals.[7] When a problem involves two or more random signals, whose statistics are time varying, it is important to preserve the phase relationships among the signals. For example, the forces on the blades of a propeller have a phase relationship that must be preserved. A loss of phase relationship makes the model unrealistic and causes the model to either underestimate or overestimate the statistics of the resultant signal. Consider a structure for which the applied loads tend to become large at the same instant. If we lose the phase relationship between the loads, then we are likely to underestimate the resultant of these loads. By a simple example we show that if $P(t)$ and $Q(t)$ are two CS processes stationarization of the superposition of these signal will lead to a loss of the phase relationship between the two signals. Let $P(t) = A(t)\sin(\omega_1 t - \psi)$, and $Q(t) = B(t)\sin(\omega_2 t)$, where $A(t)$ and $B(t)$ are zero-mean stationary random process and ψ is a fixed phase angle. It can be seen that the resultant random process $P(t) + Q(t)$ is CS. Let $S(t)$ be the resultant process after introducing two random phase angles ϕ_1 and ϕ_2 that are uniformly distributed between 0 and 2π. Then

$$S(t) = A(t)\sin(\omega_1 t - \psi + \phi_1) + B(t)\sin(\omega_2 t + \phi_2) \tag{1}$$

Figure 6 shows the autocorrelation of $S(t)$, when ϕ_1 and ϕ_2 are assumed equal. In this case the phase relationship is preserved as both the processes are shifted by the same phase angle. Figure 6 shows that the resultant process is still CS because $R_{ss}(t_2, t_1 - t_2)$ is periodic with t_2. Figure 7 shows the autocorrelation of $S(t)$, when ϕ_1 and ϕ_2 are assumed statically independent. In this case the resultant process is stationary, but the phase relationship is lost. Moreover, introducing a random phase angle changes the type of the probability density function of the random process and complicates failure analysis. If a CS process is Gaussian, the process after introducing a random phase angle is no longer Gaussian. This complicates the estimation of the probabilities of first excursion failure and fatigue failure of a system whose response is converted into a stationary process by introducing a random phase angle because equations for Gaussian random processes that are commonly used in failure analysis are no longer applicable.

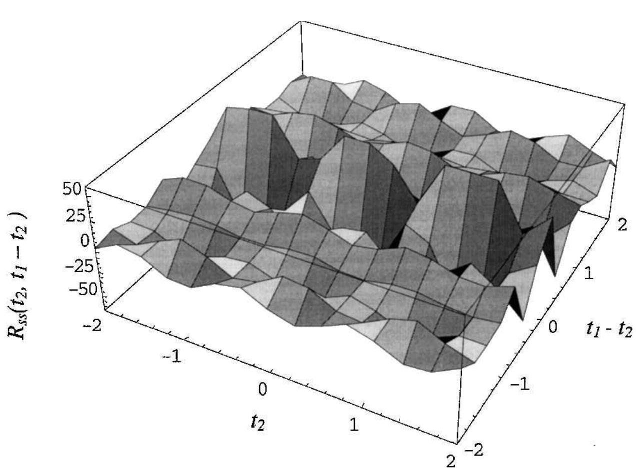

Fig. 6 Autocorrelation function of $S(t)$, when ϕ_1 and ϕ_2 are equal.

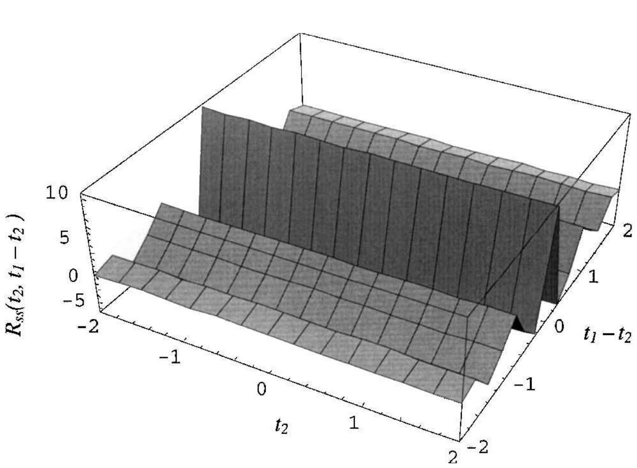

Fig. 7 Autocorrelation of $S(t)$, when ϕ_1 and ϕ_2 are statistically independent.

Stationarization causes underestimation of the risk of failure. In many design problems, the variation of the statistical properties in time is important. As a result, stationarization may give erroneous estimates of the design load or the load parameters. In general, a CS model is more conservative and realistic than a stationary approximation of a CS process obtained by the preceding methods. Nikolaidis et al.[9] showed that a CS process is more likely to cross a certain level over a period than a stationary process that has the same average intensity of fluctuation. Therefore, a structure subjected to a CS excitation has higher first excursion and fatigue failure probabilities[10] than an identical structure subjected to a stationary excitation with the same time average rms.

As mentioned earlier, in mechanics there have been a few studies on CS random processes. To the best of the authors' knowledge, Bolotin and Elishakov[11] were the first to present periodic, nonstationary processes in the literature in mechanics using a random periodic function to model loads on a cylindrical or a spherical elastic shell. Nikolaidis et al.[12] calculated the statistics of the responses of marine diesel engine shafting systems caused by CS engine and propeller excitation. George et al.[13] considered CS models for the excitation of bladed-disk assemblies rotating in nonuniform flowfields. Koenig et al.[14] modeled the pressure caused by the combustion in the cylinders of a car engine as a CS process. Lyridis et al.[15] examined CS models of the vibratory loads on an internal combustion engine. Vibrations in rotating machinery are produced by a combination of periodic and random processes. The combinations of such components result in a signal that has periodically time-varying ensemble statistics. McCormick and Nandi[16] modeled this signal as a CS process. However, these studies do not provide a methodology to find the statistics of the response of a dynamic system driven by a general CS process model.

This paper first describes the basic concepts of a CS process and the input-output relationship for the statistical properties for a multi-input multi-output (MIMO), linear-time-invariant (LTI) system under CS excitation. We have found that, under certain assumptions, the excitation on a road vehicle as a result of a rough road is a CS process. To the best of the authors' knowledge, no study on random vibration analysis of road vehicles has modeled the excitations caused of the road roughness as CS processes. We apply the methodology developed for the general MIMO, LTI system under CS excitations to the road-vehicle problem. For verification we compare the results obtained by the formulation just mentioned with the results obtained by the Monte Carlo (MC) simulation. Finally, we demonstrate the significance of a CS process model by comparing the statistics of the vehicle responses obtained by stationary and CS models.

Characterization of CS Processes

Here, we present the concepts and definitions of a CS process.[17] The autocorrelation of a CS excitation is a function of both time $t\ (=t_2)$ and the time difference $\tau = t_1 - t_2$ of two instances t_2 and t_1, when the autocorrelation is measured. Therefore, a CS process is nonstationary. We can calculate the Fourier series of the correlation, which is periodic in t, and treat each coefficient of the series as the correlation of a stationary process. This leads to two important results:

1) We can derive a relationship between the statistics of the input and output of a LTI system, which is simpler than that for a general nonstationary process.

2) We can extend the concept of spectral density used for stationary processes to CS processes.

In the following equations an upper-case character denotes a random variable or process and a lower-case character denotes realization of the variable or the process. Bold letters denote vector/matrices and nonbold letters denote scalar quantities.

We define a vector of random processes $\boldsymbol{Y}(t)$ as a first-order CS process if its probability density function is periodic with time:

$$\boldsymbol{f}_{Y(t)}(\boldsymbol{y}) = \boldsymbol{f}_{Y(t+nT)}(\boldsymbol{y}), \qquad n \in I \tag{2}$$

This implies that the mean $\eta(t)$ of a CS process is also periodic:

$$\eta(t) = E[\boldsymbol{Y}(t)] = E[\boldsymbol{Y}(t+nT)] = \eta(t+nT) \tag{3}$$

where $E[\boldsymbol{Y}(t)]$ denotes the expected value of $\boldsymbol{Y}(t)$.

Similarly, we define that a random process $\boldsymbol{Y}(t)$ is second-order CS if its joint probability density function $f_{Y(t_1),Y(t_2)}(\boldsymbol{y}_1,\boldsymbol{y}_2)$ is invariant to a shift of the time origin by an integral multiple of a constant T called time period:

$$f_{Y(t_1),Y(t_2)}(\boldsymbol{y}_1,\boldsymbol{y}_2) = f_{Y(t_1+nT),Y(t_2+nT)}(\boldsymbol{y}_1,\boldsymbol{y}_2) \tag{4}$$

Second-order cyclostationarity implies that the correlation matrix $\boldsymbol{R}_{YY}(t_1,t_2)$ of $\boldsymbol{Y}(t)$ is periodic in time:

$$\boldsymbol{R}_{YY}(t_1,t_2) = \boldsymbol{R}_{YY}(t_1+nT, t_2+nT) \tag{5}$$

The correlation matrix can be also written in the following form:

$$\boldsymbol{R}_{YY}(t+\tau,t) = \boldsymbol{R}_{YY}(t+\tau+nT, t+nT)$$

$$\text{where } t = t_2, \quad \tau = t_1 - t_2 \tag{6}$$

$\boldsymbol{R}_{YY}(t+\tau,t)$ is an $m\times m$ matrix with periodicity (period $=T$) in t, and hence, we can represent it by the Fourier series:

$$\boldsymbol{R}_{YY}(t+\tau,t) = \sum_{n=-\infty}^{\infty} \boldsymbol{r}_{YY}(n,\tau)\exp\left(\frac{j2\pi nt}{T}\right) \tag{7}$$

The coefficients of the Fourier series are

$$\boldsymbol{r}_{YY}(n,\tau) = 1/T\int_0^T \boldsymbol{R}_{YY}(t+\tau,t)\exp\left(\frac{-j2\pi nt}{T}\right)\mathrm{d}t \tag{8}$$

If a process satisfies Eqs. (3) and (5), it is called wide-sense cyclostationary (WSCS).

The Fourier transform of the cyclic correlation matrix $\boldsymbol{r}_{YY}(n,\tau)$, scaled by $1/(2\pi)$, is called *matrix of cyclic cross-spectral density* of the vector process $\boldsymbol{Y}(t)$, and it is given by the equation

$$\boldsymbol{S}_{YY}(n,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{r}_{YY}(n,\tau)\exp(-j\omega\tau)\,\mathrm{d}\tau \tag{9}$$

where

$$\boldsymbol{r}_{YY}(n,\tau) = \int_{-\infty}^{\infty} \boldsymbol{S}_{YY}(n,\omega)\exp(j\omega\tau)\,\mathrm{d}\omega \tag{10}$$

Input-Output Problem

Here we derive a generic approach for finding the response of linear system subjected to CS excitations.

Let $\boldsymbol{Y}(t)$, size $m\times 1$, be the input vector of a MIMO, LTI system with m inputs and p outputs. The output vector $\boldsymbol{Z}(t)$, size $p\times 1$, is the convolution of the input vector with the matrix of the impulse response functions $\boldsymbol{h}(t)$ (Ref. 18):

$$\boldsymbol{Z}(t) = \int_{-\infty}^{\infty} \boldsymbol{h}(t-u)\boldsymbol{Y}(u)\,\mathrm{d}u \tag{11}$$

The first- and second-order statistical properties of the output can be written in matrix notation[18]:

$$E[\boldsymbol{Z}(t)] = \int_{-\infty}^{\infty} \boldsymbol{h}(t-\tau)E[\boldsymbol{Y}(\tau)]\,\mathrm{d}\tau \tag{12}$$

$$\boldsymbol{K}_{ZZ}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{h}(t_1-\tau_1)\boldsymbol{K}_{YY}(\tau_1,\tau_2)\boldsymbol{h}'(t_2-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{13}$$

A prime denotes a transpose of a matrix or vector. Without any loss of generality, we assume that the mean values of the excitations are zero. Then, the mean of the response is also zero [Eq. (12)]. In this case the covariance matrix is equal to the correlation matrix. Now we can write an equation for correlation matrix $\boldsymbol{R}_{ZZ}(t_1,t_2)$ of the output vector $\boldsymbol{Z}(t)$ similar to Eq. (13), i.e.,

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{h}(t_1-\tau_1)\boldsymbol{R}_{YY}(\tau_1,\tau_2)\boldsymbol{h}'(t_2-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{14}$$

Putting the Fourier-series expansion of the correlation matrix $\boldsymbol{R}_{YY}$ from Eq. (7) into Eq. (14), we get

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{h}(t_1-\tau_1)\left[\sum_{n=-\infty}^{\infty}\boldsymbol{r}_{YY}(n,\tau_1-\tau_2)\right.$$

$$\left.\times\exp\left(\frac{j2\pi n\tau_2}{T}\right)\right]\boldsymbol{h}'(t_2-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{15}$$

Substituting the expression for $\boldsymbol{r}_{YY}$ from Eq. (10) in the preceding equation and rearranging the summation and integration signs yields

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{h}(t_1-\tau_1)\boldsymbol{S}_{YY}(n,\omega)$$

$$\times\exp[j\omega(\tau_1-\tau_2)]\exp\left(\frac{j2\pi n\tau_2}{T}\right)\boldsymbol{h}'(t_2-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\omega \tag{16}$$

To carry out the preceding integration, we group the functions that depend on τ_1 and τ_2 separately and adjust the exponential terms accordingly, i.e.,

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} \boldsymbol{h}(t_1-\tau_1)\exp[-j\omega(t_1-\tau_1)]\,\mathrm{d}\tau_1\right\}$$

$$\times\left\{\boldsymbol{S}_{YY}(n,\omega)\exp[j\omega(t_1-t_2)]\exp\left(\frac{j2\pi nt_2}{T}\right)\right\}$$

$$\times\left\{\int_{-\infty}^{\infty}\boldsymbol{h}'(t_2-\tau_2)\exp\left[j\left(\omega-\frac{2\pi n}{T}\right)(t_2-\tau_2)\right]\mathrm{d}\tau_2\right\}\mathrm{d}\omega \tag{17}$$

By replacing the Fourier transform of the matrix of impulse response function $\boldsymbol{h}(t)$ with the matrix of frequency response function $\boldsymbol{H}(\omega)$, we obtain

$$\boldsymbol{R}_{ZZ}[t_1,t_2] = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{H}(\omega)\boldsymbol{S}_{YY}(n,\omega)\boldsymbol{H}^*\left(\omega-\frac{2\pi n}{T}\right)$$

$$\times\exp[j\omega(t_1-t_2)]\exp\left(\frac{j2\pi nt_2}{T}\right)\mathrm{d}\omega \tag{18}$$

Substituting the factor $\boldsymbol{H}(\omega)\boldsymbol{S}_{YY}(n,\omega)\boldsymbol{H}^*(\omega-2n\pi/T)$ with $\boldsymbol{S}_{ZZ}(n,\omega)$ in the preceding expression, we get the following expression:

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{S}_{ZZ}(n,\omega)\exp[j\omega(t_1-t_2)]\,\mathrm{d}\omega$$

$$\times\exp\left(\frac{j2\pi nt_2}{T}\right) \tag{19}$$

This implies

$$\boldsymbol{R}_{ZZ}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\boldsymbol{r}_{ZZ}(n,t_1-t_2)\exp\left(\frac{j2\pi nt_2}{T}\right) \tag{20}$$

where we can write an equation, similar to Eq. (10), for the cyclic correlation matrix of the response vector $\boldsymbol{r}_{ZZ}(n,t_1-t_2)$:

$$\boldsymbol{r}_{ZZ}(n,t_1-t_2) = \int_{-\infty}^{\infty}\boldsymbol{S}_{ZZ}(n,\omega)\exp[j\omega(t_1-t_2)]\,\mathrm{d}\omega \tag{21}$$

$S_{ZZ}(n, \omega)$ in Eq. (21) is given by

$$S_{ZZ}(n, \omega) = H(\omega) S_{YY}(n, \omega) H^*(\omega - 2n\pi/T) \tag{22}$$

$R_{ZZ}(t_1, t_2)$ and $S_{ZZ}(n, \omega)$ are the correlation matrix and the matrix of cyclic cross-spectral density of the output vector, respectively. $S_{ZZ}(n, \omega)$ is the Fourier transform of the cyclic correlation matrix $r_{ZZ}(n, \tau)$. $H(\omega)$ is the matrix of the frequency response functions, and $H^*(\omega)$ is $H(\omega)$ Hermitian. It can be seen from Eq. (20) that the response is CS.

The rms (or standard deviation) of the response is an important measure of the intensity of the random fluctuation. To calculate it, we set $t_1 = t_2$ in Eqs. (20) and (21).

Stationary Process as a Special Case of CS Process

The statistics of a stationary process are only functions of the time lag $t_1 - t_2$. For a stationary process the mean vector is constant, and the correlation matrix depends only upon the time difference between the two points where the correlation is measured. From Eq. (7) it can be seen that, as a result of nonzero terms corresponding to $n \neq 0$ in the Fourier-series expansion, the correlation matrix depends on time. Hence, for a vector of stationary random processes, a cyclic correlation matrix will be a null matrix for $n \neq 0$. Dropping all the terms with $n \neq 0$ in Eqs. (20–22), we can write

$$R^S_{ZZ}(t_1, t_2) = r_{ZZ}(0, t_1 - t_2) \tag{20a}$$

$$r_{ZZ}(0, t_1 - t_2) = \int_{-\infty}^{\infty} S_{ZZ}(0, \omega) \exp[j\omega(t_1 - t_2)]\, d\omega \tag{21a}$$

$$S_{ZZ}(0, \omega) = H(\omega) S_{YY}(0, \omega) H^*(\omega) \tag{22a}$$

$R^S_{ZZ}[t_1, t_2]$ is the correlation matrix of a vector of stationary random processes. It has only one term in its Fourier-series representation. Dropping the terms corresponding to $n \neq 0$ in Eq. (20a) is equivalent to averaging of the correlation over time. To see this, put $n = 0$ in Eq. (7). Hence the stationary model of a CS process is only an approximation. Equations (20a) and (22a) give the second-order statistics of the response for a MIMO, LTI system under stationary excitations. An approach using Eqs. (20a) and (22a) will be referred in this paper as an *approximate stationary model.*

The method just mentioned developed for CS and stationary models will be called *analytical method.*

Response of a Road Vehicle Driven on a Road Made of Concrete Slabs

In real life the road roughness that a vehicle encounters is often nonstationary. Some real life nonstationary loads, such as the loads on a car passing over a rail track, are far more likely to cause damage than stationary loads with the same average intensity of fluctuation. Under certain conditions, which are frequent in real life, the road excitation on a vehicle is CS. Here are two examples:

1) Figure 8 shows the vertical front suspension load for 10 laps of an all-terrain vehicle over a test track.[19] The figure illustrates that the statistics of the load will not be a constant, but periodic in time. As explained earlier, this type of excitation is CS.

2) This section will show that the road excitation on a vehicle traveling on a road made of concrete slabs of fixed length is also CS.

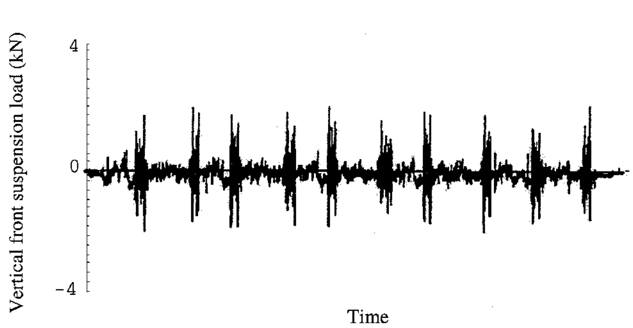

Fig. 8 Ten laps of a test track.

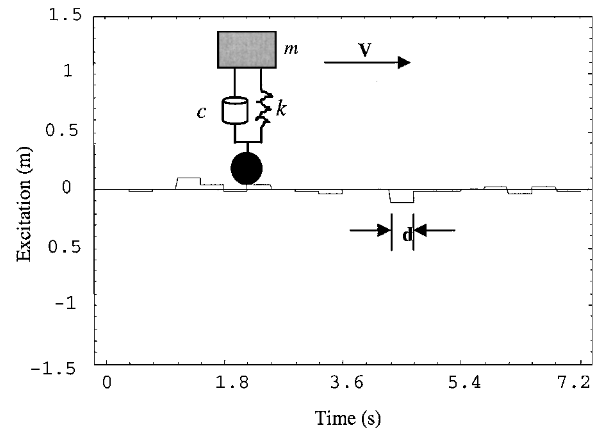

Fig. 9 Sample path of the road and vehicle model.

First, we make a few assumptions to develop a mathematical model of the road roughness. Then, we apply the analytical method, presented in the preceding two sections, to a vehicle moving on such road. We assume that the road is made of concrete slabs of constant length d, but different heights H_i. A quarter of the vehicle is modeled by a single-degree-of-frædom spring-mass-damper system. The random excitation is caused by the uneven surface of a road. The heights of slabs are assumed normally distributed, statistically independent random variables. The mean value of H_i is zero, and its standard deviation is σ_H. We also assume that the surface of each slab is horizontal. Figure 9 shows a sample path of the road with the vehicle model. In practice, the slab length may also be random. This will introduce randomness in the time period of the vehicle excitation and may influence the structure's stability and dynamics.[5] This can be particularly important if the frequency of excitation corresponding to the distance of the slabs is close to the system natural frequency. We do not account for randomness in the slab length.

Let $Y(x)$ be the height of the road at a distance x from origin. $Y(x)$ can be written as

$$Y(x) = \sum_{i=-\infty}^{\infty} H_i \,\text{pulse}(x - id, d) \tag{23}$$

where

$$\text{pulse}(x, d) = \begin{cases} 1 & \text{if} \quad 0 \leq x \prec d \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

where x is the location where the pulse begins and d is the duration of the pulse.

$Y(x)$ is a Gaussian process so that its mean and autocorrelation are sufficient to describe it. The mean of $Y(x)$ is zero. The autocorrelation of $Y(x)$ is

$$R_{YY}(x + \Delta, x) = E[Y(x + \Delta)Y(x)]$$

$$= E\left[\sum_{i=-\infty}^{\infty} H_i \,\text{pulse}(x - id + \Delta, d) \times \sum_{j=-\infty}^{\infty} H_j \,\text{pulse}(x - jd, d)\right] \tag{25}$$

where Δ is the distance between the two points at which autocorrelation is measured. The terms that correspond to $i \neq j$ in the expression for $R_{YY}(x + \Delta, x)$ are zero because the heights of the slabs are statistically independent with zero means; therefore,

$$R_{YY}(x + \Delta, x) = \sigma_H^2 \sum_{i=-\infty}^{\infty} \text{pulse}(x + \Delta - id, d)\text{pulse}(x - id, d) \tag{26}$$

Table 1 Values of parameters for the example

Parameter	Value
Damping coefficient c, kg/s	6,610.46
Slab length d, m	10
Spring coefficient k, N/m	11,433.3
Mass m, kg	1,300
Velocity V, km/h	100
Standard deviation σ_h, m	0.04

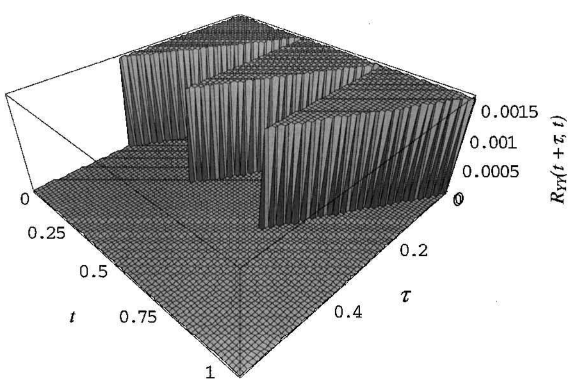

Fig. 10 Autocorrelation of the excitation.

Let the speed of vehicle be V. Then $x = Vt$. Hence the autocorrelation in the time domain is

$$R_{YY}(t+\tau, t) = \sigma_H^2 \sum_{i=-\infty}^{\infty} \text{pulse}(t+\tau-iT, T)\text{pulse}(t-iT, T) \tag{27}$$

where the time lag τ equals Δ / V and the time period T equals d / V. The expression just mentioned for the autocorrelation can be further simplified to

$$R_{YY}(t+\tau, t) = \begin{cases} \sigma_H^2 \sum_{i=-\infty}^{\infty} \text{pulse}(t-iT+\tau, T+\tau) & \text{for} \quad \tau < 0 \\ \sigma_H^2 \sum_{i=-\infty}^{\infty} \text{pulse}(t-iT, T-\tau) & \text{for} \quad \tau \geq 0 \end{cases} \tag{28}$$

Figure 10 shows the variation of the autocorrelation of the road excitation with time t, where the first measurement is taken, and the time difference τ between the first and second measurements. It can be seen that the autocorrelation is periodic in time. Therefore, the excitation caused by the road roughness is a WSCS process.

From the single-degree-of-freedom model of the vehicle, we obtain the frequency response function $H(\omega)$. Using Eqs. (8) and (28), we obtain the cyclic autocorrelation $r_{YY}(n, \tau)$ of the excitation $Y(t)$. From Eq. (9) we get the cyclic spectral density $S_{YY}(n, \omega)$ of the excitation. Then following the analytical method, Eqs. (20–22), we calculate the autocorrelation and spectral density function of the response. We took only 11 terms in the Fourier-series expansion of the autocorrelation functions of excitation and response. Table 1 shows the values of the parameters of the model in Fig. 9.

To validate the result obtained from the analytical method, we calculated the standard deviation (rms) of the response by MC simulation. We generated 20 sample paths of the road. For each sample path of the road, we calculated the response of vehicle. Then we calculated the rms of the response as a function of time based on the ensemble and compared it to its counterpart from the analytical method.

Figure 11 shows the standard deviations of the response as a function of time calculated using the MC simulation and the analytical method. The difference in two results, which is negligible, is a result of the following: 1) the error associated with numerical integration Eq. (21); 2) the finite number of terms [Eq. (11)] used in the Fourier

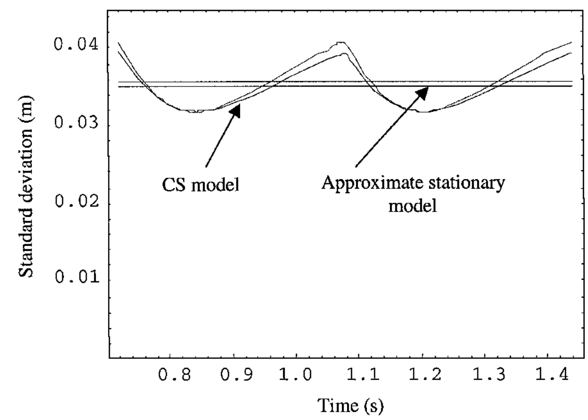

Fig. 11 Comparison of results of a CS model and approximate stationary model obtained using the analytical method (upper curve) and MC simulation (lower curve).

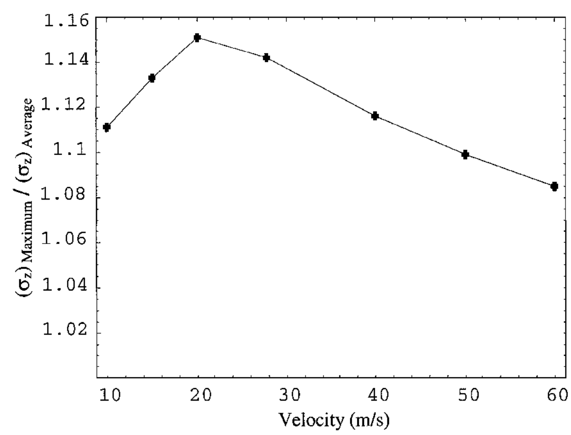

Fig. 12 Effect of velocity on the ratio of maximum standard deviation to average standard deviation.

representation of the correlation function in Eq. (7); and 3) the finite number of samples [Eq. (20)] of the response used in the MC simulation.

It is important to assess the difference between the results of the CS model [Eqs. (20–22)] and the approximate stationary model [Eqs. (20a–22a)]. Figure 11 compares the standard deviation of the response calculated using the CS model and an approximate stationary model. It can be seen from this figure that the standard deviation of the response is not constant in time but varies periodically. However, the approximate stationary model shows a constant standard deviation of the response. Figures 11 and 12 demonstrate that an approximate stationary model can underestimate the maximum value of the standard deviation of the vehicle response. At a velocity of 25 m/s, where the effect is most prominent, the maximum standard deviation $(\sigma_Z)_{\text{maximum}}$ is around 15% higher than the average standard deviation $(\sigma_Z)_{\text{average}}$ (Fig. 12). This demonstrates that an approximate stationary model can underestimate the maximum value of the standard deviation of the vehicle response. As a result, an approximate stationary model could considerably underestimate the probability of failure because of first excursion or fatigue.[10] The reasons are that 1) the first excursion probability is sensitive to the standard deviation of the stress and 2) fatigue damage is proportional to a power (which can be greater than 5) of the stress or strain at a point.

The vehicle traveling with velocity V encounters each slab for time d / V. As a result, the statistical properties are periodic in time with period d / V. This can be also seen from Fig. 11, which shows that the period of the variation of the standard deviation is 0.36 s. The inverse of this time period, scaled by 2π, is equal to the fundamental frequency ω_e of the autocorrelation of the excitation. Figure 13 shows the effect of ω_e on the standard deviation of the response.

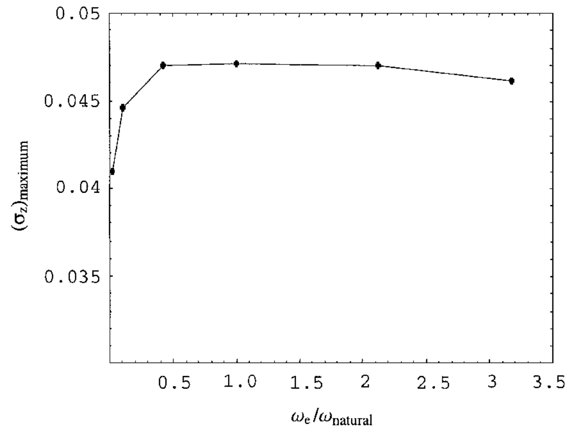

Fig. 13 Plot of maximum standard deviation of response vs ω_e/ω_n.

It is observed that when ω_e equals the natural frequency the standard deviation of response is maximum. The standard deviation is sensitive to ω_e for values less than the natural frequency, whereas it is insensitive to ω_e for values larger than the natural frequency. This information can help in vehicle design.

Dimentberg[5] studied the effect of random variability in the period of a periodically nonstationary process on the stability of a dynamic system. He concluded that, if the period of the excitation is close to the natural period of the system, even a small variability in the period (e.g., 0.5%) can improve the stability of the system and can drastically reduce the intensity of vibration of the system. Consider the example of the coal-mine cage traveling on a periodically supported cable, mentioned in the first section. Even small fluctuations in the distance between the adjacent supports can considerably reduce the intensity of the response of the cage and improve its stability. Therefore, one may have to account for the variation in the period in these cases.

In this study the natural frequency of the model of the road vehicle is 2.97 rad/s, whereas the frequency of excitation is 17.45 rad/s in Fig. 11. Therefore, we do not expect that the amplitude of vibration of the system would be sensitive to variability in the frequency of excitation, which can arise from a small variability in the lengths of the slabs. The frequency of excitation varied from 6.28 rad/s to 37.7 rad/s in Fig. 12. Again variability in the lengths of the slabs is probably not significant. This variability can be significant, however, for the results in Fig. 13 because the system natural frequency is close to the frequency of excitation in some cases.

Conclusions

This paper presented a class of random process, called cyclostationary, which can represent the random excitation on a wide class of structural systems, such as turbine blades, submarine propellers, and internal combustion engines. A key characteristic is that the statistics are not constant in time, as a stationary model would assume, but they vary periodically in time. The paper showed that the road excitation on a vehicle could be modeled as a cyclostationary process. Using a quarter-vehicle model, the paper showed that a cyclostationary model of the road excitation could provide considerably more accurate estimates of the response and reliability than a traditional stationary model.

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